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LETTER TO THE EDITOR

Hidden symmetry of the differential calculus on the quantum matrix space

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Abstract. A standard bicovariant differential calculus on the quantum matrix space $\mathrm{Mat}(m,n)_q$ is considered. Our main result is proving that the $U_q\mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$-module differential algebra $\Omega^*(\mathrm{Mat}(m,n))_q$ is in fact a $U_q\mathfrak{sl}(m+n)$-module differential algebra.

1. This work solves a problem whose simple special case occurs in the construction of a quantum unit ball of $\mathbb{C}^n$ (in the spirit of [10]). Within the framework of that theory, the automorphism group of the ball $SU(n,1) \subset SL(n+1)$ is essential. The problem is that the Wess–Zumino differential calculus in quantum $\mathbb{C}^n$ [11] seems at first glance to be only $U_q\mathfrak{sl}_n$-invariant. In that particular case the lost $U_q\mathfrak{sl}_{m+n}$-symmetry can easily be detected. The main result of this work is disclosing the hidden $U_q\mathfrak{sl}_n$-symmetry for bicovariant differential calculus in the quantum matrix space $\mathrm{Mat}\,(m,n)$. (Note that for $n=1$ we have the case of a ball).

2. We start with recalling the definition of the Hopf algebra $U_q\mathfrak{sl}_N$, $N>1$, over the field $\mathbb{C}(q)$ of rational functions of an indeterminate q [4, 5]. (We follow the notation of [3]).

For $i, j \in \{1,\ldots,N-1\}$ let

$$a_{ij} = \begin{cases} 2 & i-j=0 \\ -1 & |i-j|=1 \\ 0 & |i-j|>1. \end{cases}$$

The algebra $U_q\mathfrak{sl}_N$ is defined by the generators $\{E_i,\ F_i,\ K_i,\ K_i^{-1}\}$ and the relations

$$K_iK_j = K_jK_i \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$K_iE_j = q^{a_{ij}}E_jK_i \qquad K_iF_j = q^{-a_{ij}}F_jK_i$$

$$E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})/(q-q^{-1})$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \qquad |i-j|=1$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \qquad |i-j|=1$$

$$[E_i,E_j] = [F_i,F_j] = 0 \qquad |i-j| \neq 1.$$

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A comultiplication Δ, an antipode S and a counit ε are defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i \qquad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

$$\Delta K_i = K_i \otimes K_i \qquad S(E_i) = -K_i^{-1} E_i$$

$$S(F_i) = -F_i K_i \qquad S(K_i) = K_i^{-1}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0 \qquad \varepsilon(K_i) = 1.$$

3. Recall a description of a differential algebra $\Omega^*(\mathrm{Mat}\,(m,n))_q$ on a quantum matrix space [2, 8].

Let $i, j, i', j' \in \{1, 2, \ldots, m+n\}$, and

$$\check{R}_{ij}^{i'j'} = \begin{cases} q^{-1} & i = j = i' = j' \\ 1 & i' = j,\ j' = i \text{ and } i \neq j \\ q^{-1} - q & i = i',\ j = j' \text{ and } i < j \\ 0 & \text{otherwise.} \end{cases}$$

$\Omega^*(\mathrm{Mat}\,(m,n))_q$ is given by the generators $\{t_a^\alpha\}$ and the relations

$$\sum_{\gamma,\delta} \check{R}_{\gamma\delta}^{\alpha\beta} t_a^\gamma t_b^\delta = \sum_{c,d} \check{R}_{ab}^{cd} t_d^\beta t_c^\alpha$$

$$\sum_{a',b',\gamma',\delta'} \check{R}_{\gamma'\delta'}^{\alpha\beta} \check{R}_{ab}^{a'b'} t_{a'}^{\gamma'} dt_{b'}^{\delta'} = dt_a^\alpha t_b^\beta$$

$$\sum_{a',b',\gamma',\delta'} \check{R}_{\gamma'\delta'}^{\alpha\beta} \check{R}_{ab}^{a'b'} dt_{a'}^{\gamma'} dt_{b'}^{\delta'} = -dt_a^\alpha dt_b^\beta$$

$(a, b, c, d, a', b' \in \{1, \ldots, n\};\ \alpha, \beta, \gamma, \delta, \gamma', \delta' \in \{1, \ldots, m\})$.

Let us define a grading by $\deg(t_a^\alpha) = 0$, $\deg(dt_a^\alpha) = 1$. With that, $\mathbb{C}[\mathrm{Mat}\,(m,n)]_q = \Omega^0(\mathrm{Mat}\,(m,n)))_q$ will stand for a subalgebra of elements with zero degree.

4. Let A be a Hopf algebra and F an algebra with unit and an A-module the same time. F is said to be a A-module algebra [1] if the multiplication $m : F \otimes F \to F$ is a morphism of A-modules, and $1 \in F$ is an invariant (i.e. $a(f_1 f_2) = \sum_j a_j' f_1 \otimes a_j'' f_2$, $a1 = \varepsilon(a)1$ for all $a \in A$; $f_1, f_2 \in F$, with $\Delta(a) = \sum_j a_j' \otimes a_j''$).

An important example of an A-module algebra appears if one supplies A^* with the structure of an A-module: $\langle af, b\rangle = \langle f, ba\rangle$, $a, b \in A$, $f \in A^*$.

5. Our immediate goal is to furnish $\mathbb{C}[\mathrm{Mat}\,(m,n)]_q$ with a structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra via an embedding $\mathbb{C}[\mathrm{Mat}\,(m,n)]_q \hookrightarrow (U_q\mathfrak{sl}_{m+n})^*$.

Let $\{e_{ij}\}$ be a standard basis in $\mathrm{Mat}\,(m+n)$ and $\{f_{ij}\}$ the dual basis in $\mathrm{Mat}\,(m+n)^*$. Consider a natural representation π of $U_q\mathfrak{sl}_{m+n}$:

$$\pi(E_i) = e_{i\,i+1} \qquad \pi(F_i) = e_{i+1\,i} \qquad \pi(K_i) = qe_{ii} + q^{-1}e_{i+1\,i+1} + \sum_{j \neq i, i+1} e_{jj}.$$

The matrix elements $u_{ij} = f_{ij}\pi \in (U_q\mathfrak{sl}_{m+n})^*$ of the natural representation may be treated as 'coordinates' on the quantum group SL_{m+n} [4]. To construct 'coordinate' functions on a big cell of the Grassmann manifold, we need the following elements of $\mathbb{C}[\mathrm{Mat}\,(m,n)]_q$:

$$x(j_1, j_2, \ldots, j_m) = \sum_{w \in S_m} (-q)^{l(w)} u_{1 j_{w(1)}} u_{2 j_{w(2)}} \cdots u_{m j_{w(m)}},$$

with $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant m+n$, and $l(w) = \mathrm{card}\,\{(a,b) |\, a < b \text{ and } w(a) > w(b)\}$ being the 'length' of a permutation $w \in S_m$.

Proposition 1. $x(1, 2, \ldots, m)$ is invertible in $(U_q\mathfrak{sl}_{m+n})^*$, and the map

$$t_a^\alpha \mapsto x(1, 2, \ldots, m)^{-1} x(1, \ldots, \widehat{m+1-\alpha}, \ldots, m, m+a)$$

can be extended up to an embedding

$$i : \mathbb{C}[\mathrm{Mat}\,(m,n)]_q \hookrightarrow (U_q\mathfrak{sl}_{m+n})^*.$$

(here the $\widehat{\ }$ sign indicates the item in a list that should be omitted).

Proposition 1 allows one to equip $\mathbb{C}[\mathrm{Mat}\,(m,n)]_q$ with the structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra:

$$i\xi t_a^\alpha = \xi i t_a^\alpha \qquad \xi \in U_q\mathfrak{sl}_{m+n},\ a \in \{1, \ldots, n\},\ \alpha \in \{1, \ldots, m\}.$$

6. The main result of our work is the following theorem.

Theorem 1. $\Omega^*(\mathrm{Mat}\,(m,n))_q$ admits a unique structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra such that the embedding

$$i : \mathbb{C}[\mathrm{Mat}\,(m,n)]_q \hookrightarrow \Omega^*(\mathrm{Mat}\,(m,n))_q$$

and the differential

$$d : \Omega^*(\mathrm{Mat}\,(m,n))_q \to \Omega^*(\mathrm{Mat}\,(m,n))_q$$

are the morphisms of $U_q\mathfrak{sl}_{m+n}$-modules.

Remark 1. The bicovariance of the differential calculus on the quantum matrix space allows one to equip the algebra $\Omega^*(\mathrm{Mat}\,(m,n))_q$ with a structure of $U_q\mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$-module, which is compatible with multiplication in $\Omega^*(\mathrm{Mat}\,(m,n))_q$ and differential d. Theorem 1 implies that $\Omega^*(\mathrm{Mat}\,(m,n))_q$ possess an additional hidden symmetry, since $U_q\mathfrak{sl}_{m+n} \supsetneqq U_q\mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$.

Remark 2. Let $q_0 \in \mathbb{C}$ and q_0 is not a root of unity. It follows from the explicit formulae for $E_m t_a^\alpha$, $F_m t_a^\alpha$, $K_m^{\pm 1} t_a^\alpha$, $a \in \{1, \ldots, n\}$, $\alpha \in \{1, \ldots, m\}$, that the 'specialization' $\Omega^*(\mathrm{Mat}\,(m,n))_{q_0}$ is a $U_{q_0}\mathfrak{sl}_{m+n}$-module algebra.

7. Supply the algebra $U_q\mathfrak{sl}_{m+n}$ with a grading as follows:

$$\deg(K_i) = \deg(E_i) = \deg(F_i) = 0 \qquad \text{for } i \neq m$$

$$\deg(K_m) = 0 \qquad \deg(E_m) = 1 \qquad \deg(F_m) = 0.$$

The proofs of proposition 1 and theorem 1 reduce to the construction of graded $U_q\mathfrak{sl}_{m+n}$-modules which are dual respectively to the modules of functions $\Omega^0(\mathrm{Mat}\,(m,n))_q$ and that of 1-forms $\Omega^1(\mathrm{Mat}\,(m,n))_q$. The dual modules are defined by their generators and correlations. While proving the completeness of the correlation list, we implement the 'limit specialization' $q_0 = 1$ (see [3, p 416]).

The passage from the order-one differential calculus $\Omega^0(\mathrm{Mat}\,(m,n))_q \xrightarrow{d} \Omega^1(\mathrm{Mat}\,(m,n))_q$ to $\Omega^*(\mathrm{Mat}\,(m,n))_q$ is done via a universal argument described in a paper by Maltsiniotis [9]. This argument does not break $U_q\mathfrak{sl}_{m+n}$-symmetry.

8. Our approach to the construction of the order-one differential calculus is completely analogous to that of Drinfel'd [4], used initially to produce the algebra of functions on a quantum group by means of a universal enveloping algebra.

9. The space of matrices is the simplest example of an irreducible prehomogeneous vector space of parabolic type [7]. Such a space can also be associated with a pair constituted by a Dynkin diagram of a simple Lie algebra $\mathcal{G}$ and a distinguished vertex of this diagram. Our method can work as an efficient tool for producing $U_q\mathcal{G}$-invariant differential calculi on the above prehomogeneous vector spaces.

Note that $U_q\mathcal{G}$-module algebras of polynomials on quantum prehomogeneous spaces of parabolic type were considered in a recent work by Kebe [6].

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